A Second Order Statistical Analysis of the Operation of a Limiter-Phase Detector-Filter Cascade

By W. D. WYNN

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This paper presents a second-order statistical analysis for the cascade of a bandpass limiter, and ideal phase detector and a video filter. This cascade forms an important subsystem in the mathematical model of some coherent communication systems where information is transmitted by phase or frequency modulation of the carrier. We derive the autocorrelation function $R(t_1, t_2)$ *of the video filter response when the bandpass limiter input is a fixed amplitude-phase modulated carrier plus stationary gaussian noise. The video filter response is wide sense stationary for some nontrivial cases; these include biphase, single tone, and stationary gaussian noise phase modulation. For these cases, we obtain the video filter output average power spectrum as the Fourier transform of* $R(\tau)$ *for all values of the limiter input signal-to-noise power ratio. An application of the results of this paper is the performance of a FM-PM demodulator for a set of parameters characteristic of one mode of operation of the Apollo Unified S-Band communications system. We present the performance as a family of curves of subcarrier channel output signal-to-noise power ratio as functions of the limiter input signal-to-noise ratio where subcarrier phase modulation index is a parameter. The approach is similar to the analysis by Davenport of the signal-to-noise ratio transfer characteristic of an isolated bandpass limiter.*

I. INTRODUCTION

In some coherent communication systems, such as the Apollo Unified S-band system,[1] where information is transmitted by phase modulating a carrier, bandpass limiters[2] are used in the IF channels preceding the coherent demodulators. Ideally the bandpass limiter removes any amplitude modulation that might exist before the signal is demodulated.

Figure 1 shows a typical coherent phase demodulator used in such a system. This demodulator consists of a multiplication operation (a phase detector) with post-video filtering. The phase modulated signal is multiplied by a coherent carrier reference to yield a video signal containing the desired information. The signal into the limiter is usually accompanied by noise that is frequently assumed to be additive and gaussian. The presence of the noise affects the performance of the demodulator in a very complicated way because of the nonlinearity of the limiter. Thus it is difficult to evaluate the corruptive effect of the noise on the demodulated information.

One criterion of performance at points in a communication system is the signal-to-noise power ratio (S/N). For the cascade in Fig. 1, a problem of interest to the systems engineer is the video filter output S/N as a function of the input S/N to the limiter when the input noise is additive, stationary, and gaussian. The relationship is known between input and output S/N for an ideal bandpass limiter where the input is the sum of a stationary gaussian noise and a signal $P(t) \cos(\omega_c t + \phi)$ (see Ref. 2). For the analysis there, $P(t)$ is a random process and is slowly varying compared with $\cos \omega_c t$. The carrier phase ϕ is a random variable independent of $P(t)$ with a uniform distribution over $[0, 2\pi]$.

It is not possible to apply the known S/N transfer characteristic of the ideal bandpass limiter found in Ref. 2 directly to obtain the S/N transfer characteristic for the bandpass limiter-phase detector-video filter cascade. A knowledge of the form of the signal and the noise out of the bandpass limiter, and not just the S/N of this output, is necessary to determine the effect of the phase detector on the bandpass limiter response.

To obtain the cascade S/N transfer characteristic we apply the mathematical tools used in Ref. 2. The form of the signal assumed in the analysis of the cascade is $s(t) = P \cos[\omega_c + \theta(t) + \phi]$ where P is

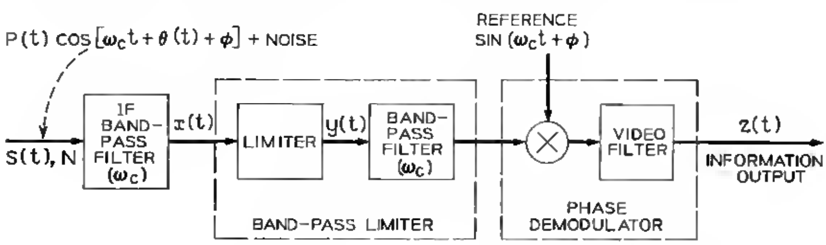

Fig. 1 — A coherent phase demodulator with IF bandpass limiting in the presence of additive noise.

a positive constant, $\theta(t)$ is phase modulation that is slowly varying compared with cos $\omega_c t$, and ϕ is a random variahle representing the arhitrary initial phase of the signal carrier. The prohahility density function of ϕ is assumed to he uniform in the interval $[0, 2\pi]$. The noise input to the handpass limiter is assumed to he additive, stationary, and gaussian with zero mean and power spectral density N. The input noise, the modulation $\theta(t)$, and the carrier phase ϕ are assumed to he jointly statistically independent. For the following analysis, the limiter is assumed to he ideal with limit level l. The transfer function of an ideal limiter is defined hy

$$y = l(x) = \begin{cases} +l, & x > 0 \\ 0, & x = 0 \\ -l, & x < 0. \end{cases} \tag{1}$$

A coherent carrier reference sin $(\omega_c t + \phi)$ is assumed to he availahle for the demodulator where ϕ is the phase of the carrier.

II. THE SECOND ORDER STATISTICAL ANALYSIS

2.1 *A Cascade Model when $s(t)$ is Narrow Band Limited*

In order to ohtain a S/N transfer characteristic for Fig. 1, the autocorrelation function of $z(t)$ is derived. When $R_z(t_1, t_2) = R_z(\tau)$ the average power spectrum of $z(t)$ is defined hy the Fourier transform of $R_z(\tau)$ and the S/N transfer characteristic can he found. An analysis of the autocorrelation function of $z(t)$ does not seem possihle for general $s(t)$. However, if the signal $s(t)$ is a narrow hand-limited process such that the handpass filters are narrow compared with the carrier frequency ω_c, the response $z(t)$ should he the same with or without the post handpass filter that precedes the phase detector. The response of the nonlinearity $l(x)$ to an input $x(t) = s(t) + n(t)$ that is narrow hand-limited ahout $\pm\omega_c$ is a family of terms narrow hand-limited ahout the frequencies $\pm n\omega_c$ where $n = 0, 1, 2, 3, \cdots$ (see equation 13-53, section 13-1 of Ref. 3). Any narrow hand-limited input to the phase detector that is not ahout $\pm\omega_c$ will generate a phase detector response ahove the cutoff frequency assumed for the video filter. For a narrow hand-limited $x(t)$ the autocorrelation function of $z(t)$ is ohtained from the analysis of Fig. 2.

2.2 *The Derivation of the Autocorrelation Function of $z(t)$*

Assume that the input $x(t)$ is narrow hand-limited such that Figs. 1 and 2 yield equivalent $z(t)$. The autocorrelation function $R_z(t_1, t_2)$ is

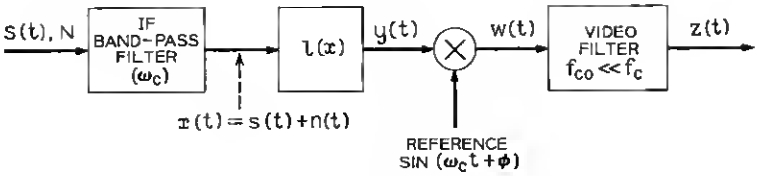

Fig. 2 — The narrow band equivalent receiver for the derivation of $R_z(t_1, t_2)$.

obtained by first deriving $R_w(t_1 , t_2)$ from the model in Fig. 2. Since z and w are related by the linear video filter, $R_z(t_1 , t_2)$ follows directly from $R_w(t_1 , t_2)$.

The Laplace transform solution of a zero memory nonlinearity with stochastic excitation is used to derive $R_w(t_1 , t_2)$ (see Chapter 13 of Ref. 3). The limiter characteristic is

$$l(x) = \frac{1}{2\pi j}\left[\int_{C+} f_+(\omega) \exp (x\omega)\, d\omega + \int_{C-} f_-(\omega) \exp (x\omega)\, d\omega\right] \tag{2}$$

where

$$f_+(\omega) = \int_0^{+\infty} l(x) \exp (-\omega x)\, dx = \frac{l}{\omega}, \qquad \text{for} \quad \text{Re}\,[\omega] > 0$$

and

$$f_-(\omega) = \int_{-\infty}^{0} l(x) \exp (-\omega x)\, dx = \frac{l}{\omega}, \qquad \text{for} \quad \text{Re}\,[\omega] < 0.$$

The variable $\omega = u + jv$ is complex with $\text{Re}[\omega] = u$. The contours C_+ and C_- are taken parallel to the v axis in the ω plane with $\text{Re}\,[\omega] > 0$ for C_+ and $\text{Re}\,[\omega] < 0$ for C_-. For convenience $l(x)$ is written symbolicly as

$$l(x) = \frac{1}{2\pi j} \int_C f(\omega) \exp (x\omega)\, d\omega \tag{3}$$

where equation (3) means the same as equation (2) when C_+ and C_- are not the same contours.

Since $w(t) = \sin (\omega_c t + \phi) \cdot l[x(t)]$, the autocorrelation function of $w(t)$ is

$$R(t_1 , t_2) = \left(\frac{1}{2\pi j}\right)^2 \int_C f(\omega_1) \int_C f(\omega_2) E\{\sin (\omega_c t_1 + \phi) \cdot \exp (\omega_1 s_1 + \omega_1 n_1)$$

$$\cdot \sin (\omega_c t_2 + \phi) \cdot \exp (\omega_2 s_2 + \omega_2 n_2)\}\, d\omega_1\, d\omega_2 \tag{4}$$

where $s_i = s(t_i)$ and $n_i = n(t_i)$, $i = 1, 2$. The order of complex integration and the expectation operation have been interchanged to get equation (4). For the assumed statistical independence of $n(t)$, $\theta(t)$, and ϕ, the expected value in equation (4) factors into

$$E\{\sin(\omega_c t_1 + \phi) \cdot \exp(\omega_1 s_1) \cdot \sin(\omega_c t_2 + \phi) \cdot \exp(\omega_2 s_2)\} \cdot \exp \tfrac{1}{2}[\sigma^2\omega_1^2 + 2R_n(\tau)\omega_1\omega_2 + \sigma^2\omega_2^2] \qquad (5)$$

where $\tau = t_2 - t_1$. The form for the cross correlation function $E\{\exp(\omega_1 n_1) \exp(\omega_2 n_2)\}$ where $n(t)$ is stationary gaussian noise has heen used in equation (5) (see pp. 476–477 of Ref. 4).

For the case where $s(t)$ is narrow hand-limited with respect to ω_c, the filter in Fig. 2 is a narrow bandpass filter, and is assumed to he symmetrical about ω_c. Then $n(t)$ can be written as (see pp. 373–374 of Ref. 4)

$$n(t) = x_c \cos \omega_c t - x_s \sin \omega_c t$$

where x_c and x_s are statistically independent stationary gaussian random processes, and

$$R_n(\tau) = R_\nu(\tau) \cos \omega_c \tau \qquad (6)$$

where $R_\nu(\tau) = R_{x_c}(\tau) = R_{x_s}(\tau)$. For a narrow bandpass IF filter, the transform of $R_\nu(\tau)$ is lowpass with a narrow handwidth compared to ω_c.

With the substitution of

$$t_1 = t, \qquad t_2 = t + \tau,$$

$$\phi^* = \phi + \omega_c t,$$

$$\sin \phi^* = \frac{\exp(j\phi^*) - \exp(-j\phi^*)}{2j},$$

and

$$\exp[R_n(\tau)\omega_1\omega_2] = \sum_{m=-\infty}^{+\infty} I_m(\omega_1\omega_2 R_\nu) \exp(jm\omega_c\tau) \qquad (7)$$

(see Article 1, Chapter 3 of Ref. 5), equation (5) hecomes

$$\begin{aligned}(-\tfrac{1}{4}) \sum_{m=-\infty}^{+\infty} & I_m(\omega_1\omega_2 R_\nu) \exp(jm\omega_c\tau) \cdot E\{[\exp(j\omega_c\tau + j2\phi^*) \\ & + \exp(-j\omega_c\tau - j2\phi^*) - \exp(j\omega_c\tau) - \exp(-j\omega_c\tau)] \\ & \cdot \exp[\omega_1 P \cos(\theta_1 + \phi^*) + \omega_2 P \cos(\theta_2 + \phi^* + \omega_c\tau)]\}. \qquad (8)\end{aligned}$$

Since exp $(j\omega_c\tau)$ and exp $[\omega_2 P \cos(\theta_2 + \phi^* + \omega_c\tau)]$ are periodic in $\omega_c\tau$ with the period 2π, the function

$$\exp(jm\omega_c\tau)E\{\ \} \tag{9}$$

in equation (8) is periodic in $\omega_c\tau$. Since $R_\nu(\tau)$ transforms to a narrow hand-limited lowpass spectrum, the autocorrelation function of $z(t)$ corresponds to tbe dc component of tbe Fourier expansion of equation (9). With the suhstitution of $\delta = \omega_c\tau$, the dc component of equation (9) is

$$\begin{aligned}
&\int_0^{2\pi} \frac{d\delta}{2\pi} \exp(jm\delta)\cdot E\{\quad\} \\
&= \sum_{r=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} I_r(\omega_1 P)\cdot I_k(\omega_2 P)\cdot E_\theta\left\{\int_0^{2\pi} \frac{d\delta}{2\pi}\right. \\
&\cdot\Big[E_{\phi^*}\{\exp[j(m+1+k)\delta + j(2+r+k)\phi^* + j(r\theta_1 + k\theta_2)]\} \\
&+ E_{\phi^*}\{\exp[j(m-1+k)\delta + j(-2+r+k)\phi^* + j(r\theta_1 + k\theta_2)]\} \\
&- E_{\phi^*}\{\exp[j(m+1+k)\delta + j(r+k)\phi^* + j(r\theta_1 + k\theta_2)]\} \\
&\left. - E_{\phi^*}\{\exp[j(m-1+k)\delta + j(r+k)\phi^* + j(r\theta_1 + k\theta_2)]\}\Big]\right\}.
\end{aligned} \tag{10}$$

Since $\phi^* = \omega_c t + \phi$, ϕ^* has a uniformly distrihuted probability density function on $[0, 2\pi]$. The averages in equation (10) witb respect to δ and ϕ^* follow. For example, the first average with respect to δ and ϕ^* is zero if $\omega + k + 1 \neq 0$ or $k + r + 2 \neq 0$, and when $k = -1 - m$ and $r = -2 - k = m - 1$ the double average is exp $[(m-1)\theta_1 - (m+1)\theta_2]$. Equation (8) reduces to

$$\begin{aligned}
(-\tfrac{1}{4}) \sum_{m=-\infty}^{+\infty} I_m(\omega_1\omega_2 R_\nu)\Big[& I_{m-1}^{(\omega_1 P)} I_{-(m+1)}^{(\omega_2 P)} E\{\exp[j(m-1)\theta_1 - j(m+1)\theta_2]\} \\
& + I_{m+1}^{(\omega_1 P)} I_{-(m-1)}^{(\omega_2 P)} E\{\exp[j(m+1)\theta_1 - j(m-1)\theta_2]\} \\
& - I_{m+1}^{(\omega_1 P)} I_{-(m+1)}^{(\omega_2 P)} E\{\exp[j(m+1)\theta_1 - j(m+1)\theta_2]\} \\
& - I_{m-1}^{(\omega_1 P)} I_{-(m-1)}^{(\omega_2 P)} E\{\exp[j(m-1)\theta_1 - j(m-1)\theta_2]\}\Big].
\end{aligned} \tag{11}$$

The terms in equation (11) for positive and negative m can he comhined by noting that $I_{-m}(x) = I_m(x)$. Witb the substitutions

$$I_m(\omega_1\omega_2 R_\nu) = \sum_{q=0}^{+\infty} \frac{\omega_1^{m+2q}\omega_2^{m+2q} R_\nu^{m+2q}}{2^{m+2q} q!\, \Gamma(m+q+1)} \tag{12}$$

and

$$h_{m,k} = \frac{1}{2\pi j} \int_C f(\omega)\omega^k \exp\left(\frac{\sigma^2\omega^2}{2}\right) I_m(\omega P)\, d\omega, \tag{13}$$

the autocorrelation function of $z(t)$ is

$$\begin{aligned} R_z(t_1, t_2) = \tfrac{1}{4} \sum_{m=0}^{\infty} \sum_{q=0}^{\infty} & \frac{\epsilon_m R_v^{2q+m}}{2^{2q+m} q!\,(q+m)!} \\ & \cdot [h^2_{m+1,2q+m} R_\theta(m+1, m+1, t_1, t_2) \\ & + h^2_{m-1,2q+m} R_\theta(m-1, m-1, t_1, t_2) \\ & - h_{m+1,2q+m} h_{m-1,2q+m} R_\theta(m+1, m-1, t_1, t_2) \\ & - h_{m+1,2q+m} h_{m-1,2q+m} R_\theta(m-1, m+1, t_1, t_2)] \end{aligned} \tag{14}$$

where

$$\epsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m > 0 \end{cases}$$

and $R_\theta(A, B, t_1, t_2) = E\{\cos[A\theta(t_1) - B\theta(t_2)]\}$ for any integers A and B.

III. THE CLOSED FORM SOLUTION FOR $h_{m,k}$

The autocorrelation function of $z(t)$ given in equation (14) contains the constants $h_{m,k}$ where $m + k$ are odd integers. For the ideal limiter characteristic of equation (1), there are closed form solutions for these parameters. Since $f_+(\omega) = l/\omega$ for Re $[\omega] > 0$ and $f_-(\omega) = l/\omega$ for Re $[\omega] < 0$, equation (13) becomes

$$h_{m,k} = \frac{1}{2\pi j} \int_{C_-} l\omega^{k-1} I_m(\omega P) \exp\left(\frac{\sigma^2\omega^2}{2}\right) d\omega + \frac{1}{2\pi j} \int_{C_+} l\omega^{k-1} I_m(\omega P) \exp\left(\frac{\sigma^2\omega^2}{2}\right) d\omega \tag{15}$$

where C_- is the contour $(-\epsilon - j\infty, -\epsilon + j\infty)$ and C_+ is the contour $(+\epsilon - j\infty, +\epsilon + j\infty)$. By the change of variable $\omega = jx$ and the substitution of $I_m(z) = (j)^{-m} J_m(jz)$, analytic continuation can be applied for $m \geqq 0$ and $k \geqq 0$ to give

$$h_{m,k} = \frac{l}{\pi} (j)^{k+m-1} \int_{-\infty}^{\infty} x^{(k-1)} J_m(xP) \exp\left[\frac{-\sigma^2 x^2}{2}\right] dx. \tag{16}$$

When $m + k$ is even, the integrand of equation (16) is odd and $h_{m,k} = 0$.

When $m + k$ is odd, the integrand of equation (16) is even and

$$h_{m,k} = \frac{2l}{\pi} (j)^{k+m-1} \cdot \frac{\Gamma\left(\frac{m+k}{2}\right)\left(\frac{P^2}{2\sigma^2}\right)^{m/2}}{2\Gamma(m+1)\left[\frac{\sigma}{(2)^{\frac{1}{2}}}\right]^k} \, {}_1F_1\left(\frac{m+k}{2}; m+1; \frac{-P^2}{2\sigma^2}\right) \tag{17}$$

where a solution has been used for the integral

$$\int_0^\infty x^{k-1} J_m(xP) \exp\left[\frac{-\sigma^2 x^2}{2}\right] dx \tag{18}$$

in terms of the confluent hypergeometric function ${}_1F_1(\alpha; \beta; -x)$ (see equation A.1.49, p. 1079 of Ref.6). For the case when m and k are non-negative integers ${}_1F_1(m + k/2; m + 1; -x)$ can be expressed in closed form in terms of first and second kind modified Bessel functions. A list of these expressions is given by Middleton (see equation A.1.31, section A 1.2 of Ref. 6). A collection of $h_{m,k}$ in closed form for low order indices is given in Table I. For Table I, $x = P^2/2\sigma^2$ is the input signal-to-noise power ratio into the limiter in Fig. (2).

Any of the $h_{m,k}$ in equation (14) can be found in closed form from Table I by using the recurrence relations

$$h_{m+2,k} = h_{m,k} - \frac{2(m+1)}{P} h_{m-1,k-1} + \frac{4(m+1)m}{P^2} h_{m,k-2} , \tag{19}$$

$$h_{m+1,k+1} = -\frac{P}{\sigma^2} h_{m,k} - \frac{(k-m-2)}{\sigma^2} h_{m-1,k-1} + \frac{2(k-m-2)m}{\sigma^2 P} h_{m,k-2} , \tag{20}$$

and

$$h_{m,k+2} = \frac{(m-k)}{\sigma^2} h_{m,k} + \frac{P^2}{\sigma^4} h_{m-2,k} - \frac{(m-k)}{\sigma^4} P h_{m-1,k-1} . \tag{21}$$

Equation (19) is derived from equation (16) by using the Bessel function identity

$$J_{m+2}(xP) = \frac{2(m+1)}{Px} J_{m+1}(xP) - J_m(xP). \tag{22}$$

Equation (20) is derived through a by-parts integration of equation (16) and the application of equation (19). Equation (21) is derived through by-parts integration of equation (16). In the development of equations (19), (20) and (21), the integral in equation (16) is re-

TABLE I — CLOSED FORM SOLUTIONS OF SOME $h_{m,k}$

m	k	$h_{m,k}$
1	0	$\frac{lP}{(2\pi)^{\frac{1}{2}}\sigma} e^{-x/2}[I_0(x/2) + I_1(x/2)]$
0	1	$\frac{(2)^{\frac{1}{2}}l}{(\pi)^{\frac{1}{2}}\sigma} e^{-x/2} I_0(x/2)$
2	1	$\frac{-(2)^{\frac{1}{2}}l}{(\pi)^{\frac{1}{2}}\sigma} e^{-x/2} I_1(x/2)$
1	2	$\frac{-lP}{(2\pi)^{\frac{1}{2}}\sigma^3} e^{-x/2}[I_0(x/2) - I_1(x/2)]$
3	2	$\frac{-lP}{(2\pi)^{\frac{1}{2}}\sigma^3} e^{-x/2}\left[I_0(x/2) - \left(1 + \frac{4}{x}\right)I_1(x/2)\right]$
0	3	$\frac{-(2)^{\frac{1}{2}}l}{(\pi)^{\frac{1}{2}}\sigma^3} e^{-x/2}[(1 - x)I_0(x/2) + xI_1(x/2)]$
2	3	$\frac{lP^2}{(2\pi)^{\frac{1}{2}}\sigma^5} e^{-x/2}\left[I_0(x/2) - \left(1 + \frac{1}{x}\right)I_1(x/2)\right]$
4	3	$\frac{-lP^2}{(2\pi)^{\frac{1}{2}}\sigma^5} e^{-x/2}\left[\left(1 + \frac{4}{x} + \frac{12}{x^2}\right)I_1(x/2) - \left(1 + \frac{3}{x}\right)I_0(x/2)\right]$
1	4	$\frac{lP}{(2\pi)^{\frac{1}{2}}\sigma^5} e^{-x/2}[(3 - 2x)I_0(x/2) + (2x - 1)I_1(x/2)]$
3	4	$\frac{-lP^3}{(2\pi)^{\frac{1}{2}}\sigma^7} e^{-x/2}\left[\left(1 + \frac{1}{2x}\right)I_0(x/2) - \left(1 + \frac{3}{2x} + \frac{2}{x^2}\right)I_1(x/2)\right]$
5	4	$\frac{-lP^3}{(2\pi)^{\frac{1}{2}}\sigma^7} e^{-x/2}\left[\left(1+\frac{9}{2x}+\frac{12}{x^2}\right)I_0(x/2)-\left(1+\frac{11}{2x}+\frac{18}{x^2}+\frac{48}{x^3}\right)I_1(x/2)\right]$

stricted to the half interval $[0, \infty)$ which is possible since the integrand of equation (16) is even when $m+k$ is odd.

IV. THE AVERAGE POWER SPECTRUM OF $z(t)$

The autocorrelation function of $z(t)$ given in equation (14) becomes time independent such that $z(t)$ has the average power spectrum $S_z(\omega) = F[R_z(\tau)]$ when $R_\theta(A, B, t_1, t_2) = R_\theta(A, B, \tau)$ for integers A and B. There are some important cases of $\theta(t)$ for which R_θ is time independent.

If θ is a bipbase modulation with $\theta(t) = \pm \mid \theta \mid$ tbat has a zero mean and autocorrelation function (see equation 9-42, section 9-2 of Ref. 4)

$$R_\theta(\tau) = \begin{cases} \mid \theta \mid^2 \left(1 - \dfrac{\mid \tau \mid}{T}\right), & \text{for} \quad \mid \tau \mid \leqq T \\ 0, \quad \text{for} \quad \mid \tau \mid > T \end{cases} \tag{23}$$

then

$$R_\theta(A, B, t_1, t_2) \\ = \cos A \mid \theta \mid \cdot \cos B \mid \theta \mid + \sin A \mid \theta \mid \cdot \sin B \mid \theta \mid \cdot r_\theta(\tau) \tag{24}$$

where $r_\theta(\tau) = R_\theta(\tau)/\mid \theta \mid^2$ is the normalized autocorrelation function of $\theta(t)$. Then R_θ is a function of $\tau = t_2 - t_1$.

For a single tone modulation given by $\theta(t) = m_1 \sin(\omega_1 t + \xi)$ where ξ is a random variable with a uniform probability density function on $[0, 2\pi]$, a simple Bessel series expansion gives

$$R_\theta(A, B, t_1, t_2) = \sum_{n=0}^{\infty} \epsilon_n J_{2n}(Am_1) J_{2n}(Bm_1) \cos(2n\omega_1\tau) \\ + \sum_{n=1}^{\infty} \epsilon_n J_{2n-1}(Am_1) J_{2n-1}(Bm_1) \cos[(2n - 1)\omega_1\tau] \tag{25}$$

where

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n > 0. \end{cases}$$

For the single tone modulation, R_θ depends only on the time difference τ. If $\theta(t)$ is the sum of tones

$$\theta(t) = \sum_{p=1}^{N} m_p \sin(\omega_p t + \xi_p) \tag{26}$$

where ξ_p, $p = 1, \cdots, N$, are independent random variables with

uniform probability density functions on [0, 2π], R_θ is again independent of time.

If $\theta(t)$ is a stationary gaussian process with zero mean, variance σ_θ^2 and autocorrelation function $K_\theta(\tau)$, then $R_\theta(A, B, t_1, t_2) = R_\theta(A, B, \tau)$. The second-order characteristic function for the stationary gaussian process is defined as (see equation 112, Chapter 7 of Ref. 4)

$$\begin{aligned} \Phi_\theta(\omega_1, \omega_2; \tau) &= E(\exp\{j[\omega_1\theta(t+\tau) + \omega_2\theta(t)]\}) \\ &= \exp[-\tfrac{1}{2}K_\theta(0)(\omega_1^2 + \omega_2^2) - K_\theta(\tau)\omega_1\omega_2]. \end{aligned} \tag{27}$$

Then

$$\begin{aligned} R_\theta(A, B, t_1, t_2) &= \text{Real Part } E\{\exp(jA\theta_1 - jB\theta_2)\} \\ &= \exp\left[-\frac{\sigma_\theta^2}{2}(A^2 + B^2)\right]\cdot \exp[ABK_\theta(\tau)] \\ &= R_\theta(A, B, \tau). \end{aligned} \tag{28}$$

The validity of equation (14) depends on the narrow band-limited assumption for the modulated signal $s(t)$ at the carrier frequency w_c. For $s(t)$ to he narrowhand limited, the parameter values that the modulation functions can have are restricted.

V. AN APPLICATION OF THE R_z RESULTS TO THE PERFORMANCE OF A SUBCARRIER CHANNEL

A modulation technique sometimes used for communication is FM-PM where the carrier is phase modulated by a subcarrier that is in turn frequency modulated by the information waveform. The FM-PM signal is of the form

$$s(t) = P\cos\{\omega_c t + \phi + m_1 \sin[\omega_1 t + \xi + \lambda(t)]\} \tag{29}$$

wbere P, ω_c, ω_1 and m_1 are constants, ϕ and ξ are independent random variables usually assumed to bave uniform probability density functions over [0, 2π], and $\lambda(t)$ is tbe integral of the information waveform. In a typical application, $\omega_c \gg \omega_1$ and $\lambda(t)$ is slowly varying compared with $\cos\omega_1 t$. With these restrictions the information $\dot{\lambda}(t)$ can be recovered from $s(t)$ with the receiver shown in Fig. 3.

The purpose of the bandpass limiter is to remove the effect of variations that might occur in P. For tbe ideal case where $s(t)$ is not perturbed by noise, the subcarrier filter input $z(t)$ is

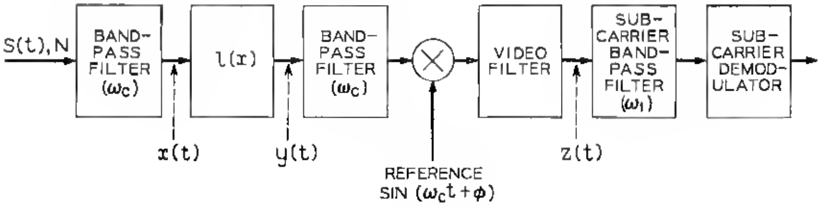

Fig. 3 — FM-PM receiver with ideal bandpass limiter.

$$
\begin{aligned}
z(t) &= -\frac{2l}{\pi} \sin \{m_1 \sin [\omega_1 t + \lambda(t) + \xi]\} \\
&= -\frac{4l}{\pi} \sum_{n=1}^{\infty} J_{2n-1}(m_1) \sin \{(2n - 1)[\omega_1 t + \lambda(t) + \xi]\}.
\end{aligned} \tag{30}
$$

If $\lambda(t)$ is slowly varying compared with $\cos \omega_1 t$, the information can be recovered with a subcarrier filter that passes only the first component of the sum in equation (30). For the noiseless case the subcarrier filter response is then

$$
-\frac{4l}{\pi} J_1(m_1) \sin [\omega_1 t + \lambda(t) + \xi]. \tag{31}
$$

After additional processing in a subcarrier demodulator, $\dot{\lambda}(t)$ is obtained from equation (31). One criterion of performance of the receiver is the S/N out of the subcarrier filter as a function of the limiter input S/N, $x = P^2/2\sigma^2$. Since $\lambda(t)$ varies slowly compared with $\cos \omega_1 t$, the output S/N for the subcarrier filter is determined with sufficient accuracy by setting $\lambda(t) \equiv 0$. If $\lambda(t) \equiv 0$, the subcarrier output S/N follows directly from equations (14) and (25). Substitution of equation (25) into equation (14) gives the power spectrum

$$
\begin{aligned}
S_z(\omega)\Big|_{\theta = m_1 \sin(\omega_1 t + \xi)} &\cong 2h_{10}^2 \sum_{n=1}^{\infty} J_{2n-1}^2(m_1) \cdot F[\cos (2n - 1)\omega_1 \tau] \\
&+ (\tfrac{1}{4})[\sigma h_{01} - \sigma h_{21} J_0(2m_1)]^2 \cdot F[r_v(\tau)] \\
&+ (\tfrac{1}{2})\sigma^2 h_{21}^2 \sum_{n=1}^{\infty} J_n^2(2m_1) \cdot F[r_v(\tau) \cdot \cos (n\omega_1 \tau)] \\
&+ (\tfrac{1}{2})\sigma^4 h_{12}^2 \sum_{n=1}^{\infty} J_{2n-1}^2(m_1) \cdot F[r_v^2(\tau) \cdot \cos (2n - 1)\omega_1 \tau] \\
&+ (\tfrac{1}{16}) \sum_{n=0}^{\infty} \epsilon_n[\sigma^2 h_{12} J_n(m_1) - \sigma^2 h_{32} J_n(3m_1)]^2 \cdot F[r_v^2(\tau) \cdot \cos n\omega_1 \tau]
\end{aligned}
$$

$$+ (\tfrac{1}{32})[\sigma^3 h_{03} - \sigma^3 h_{23} J_0(2m_1)]^2 \cdot F[r_v^3(\tau)]$$

$$+ (\tfrac{1}{16})\sigma^6 h_{23}^2 \sum_{n=1}^{\infty} J_n^2(2m_1) \cdot F[r_v^3(\tau) \cdot \cos n\omega_1 \tau]$$

$$+ (\tfrac{1}{96}) \sum_{n=0}^{\infty} \epsilon_n [\sigma^3 h_{23} J_n(2m_1) - \sigma^3 h_{43} J_n(4m_1)]^2 \cdot F[r_v^3(\tau) \cdot \cos n\omega_1 \tau]$$

$$+ (\tfrac{1}{32})\sigma^8 h_{14}^2 \sum_{n=1}^{\infty} J_{2n-1}^2(m_1) \cdot F[r_v^4(\tau) \cdot \cos (2n - 1)\omega_1 \tau]$$

$$+ (\tfrac{1}{192}) \sum_{n=0}^{\infty} \epsilon_n [\sigma^4 h_{14} J_n(m_1) - \sigma^4 h_{34} J_n(3m_1)]^2 \cdot F[r_v^4(\tau) \cdot \cos n\omega_1 \tau]$$

$$+ (\tfrac{1}{768}) \sum_{n=0}^{\infty} \epsilon_n [\sigma^4 h_{34} J_n(3m_1) - \sigma^4 h_{54} J_n(5m_1)]^2 \cdot F[r_v^4(\tau) \cdot \cos n\omega_1 \tau] \qquad (32)$$

where $r_v = R_v/R_v(0) = R_v/\sigma^2$. The approximation, equation (32), neglects all the terms of equation (14) containing the factor R_v^{2q+m} where $2q + m > 4$. The terms in equation (32) are the significant terms of $S_z(\omega)$ for the single tone modulation. The spectrum in equation (32) is the weighted sum of terms of the form

$$F[r_v^n(\tau) \cos m\omega_1 \tau] = \frac{1}{2\pi} F[r_v^n(\tau)] * F[\cos m\omega_1 \tau] \qquad (33)$$

where * is the convolution operation. Since $F[\cos m\omega_1\tau]$ is a pair of impulses of weight π at $\pm m\omega_1$,

$$F[r_v^n(\tau) \cos m\omega_1 \tau] = \tfrac{1}{2}[S_{v,n}(\omega + m\omega_1) + S_{v,n}(\omega - m\omega_1)] \qquad (34)$$

where

$$S_{v,n}(\omega) = F[r_v^n(\tau)].$$

The first term in the spectrum of equation (32) is the signal content of $z(t)$. All other terms of equation (32) correspond to noise alone or a combination of signal and noise. All terms of equation (32) except the first term are usually combined to give the interference (noise) spectrum at the output of the video filter.

A computation was made for the subcarrier filter output S/N as a function of the input S/N x. The following conditions are assumed for the computation.

(*i*) The power spectrum of the input gaussian noise to the cascade in Fig. 1 is uniform over the bandwidth of the prelimiter handpass filter.

(*ii*) The prelimiter bandpass filter is assumed to have a gaussian transfer function such that

$$r_z(\tau) = \exp\left[\frac{-\tau^2\omega_0^2}{\pi}\right]. \tag{35}$$

(*iii*) The subcarrier amplitude transfer function is

$$|H(j\omega)| = \begin{cases} 1, & \omega_1 - \dfrac{\Delta\omega}{2} < |\omega| < \omega_1 + \dfrac{\Delta\omega}{2} \\ 0, & \text{all other } \omega, \end{cases} \tag{36}$$

where $\Delta\omega \ll \omega_0$. Also, $\omega_0 = 12.566 \times 10^6$ and $\omega_1 = 6.434 \times 10^6$ are assumed. Substitution of equation (35) into equation (34) gives

$$S_{v,n}(\omega) = F[r_v^n(\tau)] = \frac{\pi}{\omega_0(n)^{\frac{1}{2}}} \exp\left[-\frac{\pi}{4n}\left(\frac{\omega}{\omega_0}\right)^2\right]. \tag{37}$$

From condition *iii*, the noise spectrum in the passband of tbe subcarrier filter is approximately constant when $\omega = \omega_1$. The signal and noise powers out of the subcarrier filter follow from $S_z(\omega_1)$. The signal power is $2h_{10}^2J_1^2(m_1)$; the noise power is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} [S_z(\omega) - 2h_{10}^2J_1^2(m_1)\cdot F(\cos\omega_1\tau)]\cdot|H(j\omega)|^2\, d\omega \cong [S_z'(\omega_1)]\cdot 2\,\Delta f$$

where Δf is the width of the subcarrier filter and where S_z' is equation (32) with the first term omitted. The function

$$S(m_1) = \frac{2h_{10}^2J_1^2(m_1)}{x[S_z'(\omega_1)]} \tag{38}$$

was computed for x between 0.01 and 100 with m_1 as a parameter. The results of the computation are shown in Fig. (4). For a given m_1 and x, the output S/N for the subcarrier filter is $x/2\Delta f\cdot S(m_1)$.

VI. SUMMARY

A general, second order statistical analysis is presented for the cascade of a narrow bandpass limiter, an ideal phase detector, and a video filter. In this analysis, the input to the limiter is assumed to be tbe sum of a stationary gaussian noise and a fixed amplitude phase modulated sine wave. Tbe autocorrelation function of the cascade response is obtained as a function of the signal-to-noise ratio x at the limiter input, tbe normalized autocorrelation function of the lowpass equivalent for tbe limiter input noise $r_z(\tau)$, and the phase modulation $\theta(t)$.

The cascade response $z(t)$ bas tbe autocorrelation function $R_z(t_1, t_2)$

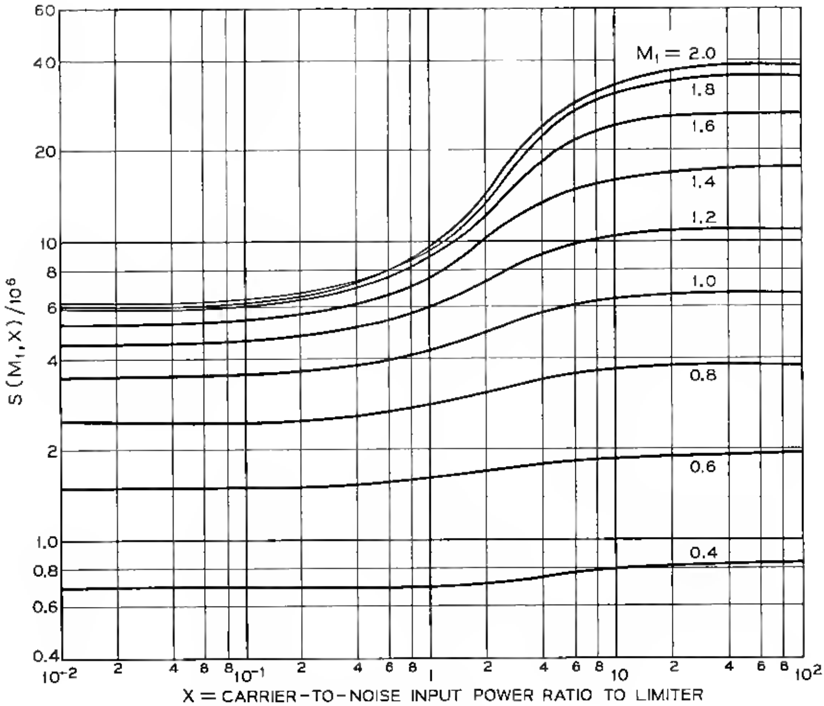

Fig. 4 — The unit bandwidth subcarrier filter output, S/N normalized by x where $10^{-2} \leqq x \leqq 10^2$ and M_1 is a parameter.

that can be time dependent. However, for some important cases of $\theta(t)$, $R_s(t_1, t_2) = R_s(\tau)$, and the cascade response has the average power spectrum $S_s(\omega) = F[R_s(\tau)]$ where F is the Fourier transform operation with respect to τ. The cases of $\theta(t)$ considered that yield $R_s(\tau)$ are the random biphase waveform $\theta = \pm | \theta |$, the single tone $\theta(t) = m_1 \sin(\omega_1 t + \xi)$, and the stationary gaussian process with autocorrelation function $K_\theta(\tau)$.

The dependence of $R_s(t_1, t_2)$ on the limiter input S/N appears in the h parameters. These parameters can be obtained in closed form as functions of the modified Bessel functions $I_0(x/2)$ and $I_1(x/2)$. The lower order h parameters encountered in the first few terms of the series for R_s are found, and recurrence relations are derived through which higher order h parameters can be derived easily.

For the modulation types that make R_s a function of τ alone, the power spectrum $S_s(\omega)$ is known for all values of the limiter input S/N x. Then the S/N can be derived in any frequency band at the output of

the video filter in Fig. 1 as a function of any S/N into the limiter.

The performance of a subcarrier channel was considered wbere $\theta(t) = m_1 \sin [\omega_1 t + \lambda(t) + \xi]$. The subcarrier was assumed to be pbase modulated by a narrowband low pass process $\lambda(t)$. The S/N at the output of the subcarrier filter was obtained by computation of tbe approximation of equation (32). For this example, a gaussian prelimiter bandpass filter was assumed. For this filter sbape, $r_v^n(\tau)$ and its transform $S_{v,n}(\omega)$ are gaussian for all integers n. Some representative parameters from the Apollo unified S-band communication system[1] were assumed. Tbese were

(*i*) A prelimiter noise equivalent bandwidtb of 4 MHz.
(*ii*) A subcarrier frequency of 1.024 MHz.
(*iii*) A subcarrier noise equivalent bandwidth of 0.2 MHz.
(*iv*) An input S/N range of $0.01 \leqq x \leqq 100$.
(*v*) A set of modulation indices $m_1 = (0.2)k$, $k = 2, 3, 4, 5, 6, 7, 8, 9, 10$.

Tbe results are given in Fig. 4. Tbe differential between subcarrier filter output S/N at low and higb values of x is a monotonically increasing function of m_1 for $0.4 \leqq m_1 \leqq 2.0$. Tbe shapes of tbe curves are similar to tbat of the $(S/N)_0/(S/N)_i$ curve obtained by Davenport.[2]

VII. ACKNOWLEDGMENT

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